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Quasi-stationary behavior for Markov-modulated Markov chains

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Abstract

In this paper, we show the quasi-stationary distribution for Markov-modulated Markov chains. We focus on two fundamental aspects (existence and uniqueness, domain of attraction) in connection with quasi-stationary distribution. We first provide a sufficient criterion for the existence of the quasi-stationary distribution. An iterative algorithm to compute all quasi-stationary distributions is presented. We then carry out a study on the domain of attraction for the quasi-stationary distribution under a uniqueness condition. In addition, we apply the results to $M/G/1$-type Markov chains, and characterize the asymptotic behavior of the quasi-stationary distribution for this model. Finally, a scalar example is given to illustrate these results.

Introduction

Consider an absorbing discrete-time homogeneous Markov chain $\{\Phi_n, n \in Z_+\}$ on a countable state space $E = \{0, 1, 2, \cdots\}$ with one-step transition matrix P. Assume that 0 is the absorbing state and $E^* = E\backslash\{0\}$ is an irreducible and aperiodic communicating class of Φ_n. By T, we denote the time of absorption, or

$$T = \inf\{n \geq 0 : \Phi_n = 0\}.$$

A *quasi-stationary distribution* for Φ_n is a probability measure $\boldsymbol{\mu}^T = (\boldsymbol{\mu}_j, j \in E^*)$ on E^* which satisfies the following equation

$$P_{\boldsymbol{\mu}^T}(X_n = j|T > n) = \boldsymbol{\mu}_j, \quad \text{for any } \; j \in E^*, \tag{1}$$

for all $n = 0, 1, \ldots$.

Quasi-stationary distributions have been used in a variety of diverse contexts for modelling the long-term behaviour of stochastic systems which, in some sense, terminate, but appear to be stationary over any reasonable time scale. In this paper, we investigate the quasi-stationary distribution for an absorbing Markov-modulated Markov chain, which is a two-dimensional Markov chain with transition probability matrix P as follows:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ P_{1,0}\boldsymbol{e} & P_{1,1} & P_{1,2} & P_{1,3} & \cdots \\ P_{2,0}\boldsymbol{e} & P_{2,1} & P_{2,2} & P_{2,3} & \cdots \\ P_{3,0}\boldsymbol{e} & P_{3,1} & P_{3,2} & P_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2}$$

where $P_{i,j}$ is a matrix of size $k_i \times k_j$ with $k_i, k_j < \infty$ for each $i, j \geq 0$, and $\boldsymbol{e}$ is a column vector of ones with an appropriate order according to the context.

The study of quasi-stationary distribution on Markov chains was initiated by Yaglom [4], first for branching processes, and later extended by many other researches (see, for example, [1], [2] and references therein). For a countable state Markov chain, the following two are among the fundamental issues for quasi-stationary distributions (see, for example, Pakes [3]):

(1) determination of all quasi-stationary distributions;

(2) description of the domain of attraction.

In this paper, we consider all two fundamental aspects, mentioned above, of the quasi-stationary distribution for an absorbing Markov-modulated Markov chain. Our main contributions are threefold:

(1) Providing sufficient and necessary conditions for the existence of a quasi-stationary distribution and a recursive scheme to calculate all quasi-stationary distributions.

(2) Characterizing the asymptotic behavior of the quasi-stationary distribution and the decay parameter of the chain. We show that there are only two types of asymptotic properties for the quasi-stationary distribution.

(3) Investigating the domain of attraction of quasi-stationary distribution. No such a study could be found in the literature for Markov-modulated Markov chains, even for quasi-birth-and-death processes. We give a sufficient condition for an initial measure to be in the domain of attraction of a quasi-stationary distribution.

Main results

A nontrivial, nonnegative row vector $\boldsymbol{\mu}^T(\beta) = (\boldsymbol{\mu}_i^T(\beta), 1 \leq i < \infty)$, where

$$\boldsymbol{\mu}_i^T(\beta) = (\boldsymbol{\mu}_{(i,j)}(\beta), 1 \leq j \leq k_i), \tag{3}$$

is called a β-invariant measure of P on E^* if the following equation holds

$$\beta\boldsymbol{\mu}^T(\beta) = \boldsymbol{\mu}^T(\beta)P. \tag{4}$$

The following relation between a quasi-stationary distribution and β-invariant distribution is well-known, which is crucial for the characterization of the quasi-stationary distribution

Lemma 0.1. *For any $\rho \leq \beta < 1$, $\boldsymbol{\mu}^T(\beta)$ is a quasi-stationary distribution for P on E^* if and only if $\boldsymbol{\mu}^T(\beta)$ is a β-invariant distribution for P on E^*.*

Now, based on the above lemma, we first give the criterion for the existence of the quasi-stationary distribution.

Theorem 0.2. *Suppose that the absorption is certain, that is $P_{(i,j)}\{T < \infty\} = 1$ for some (then all) $(i,j) \in E^*$, $\mathbf{K} = \sharp\{(i,j) \in E^* : \kappa_{(i,j)} > 0\} < \infty$ and the equation*

$$\lim_{n\to\infty, k\to\infty} \frac{\sum_{i'=k}^{\infty} \sum_{j'=1}^{m} {}_{(i'',j'')}\hat{P}_{((i_n,j_n),(i',j'))}(\alpha)\hat{P}((i',j'),(i'',j''))}{{}_{(i'',j'')}\hat{P}_{((i_n,j_n),(i'',j''))}(\alpha)} = 0, \tag{5}$$

holds for an infinite sequence of integers $\{(i_n, j_n)\}$, then a quasi-stationary distribution exists.

Secondly, we show a recursive scheme of the calculation of β-invariant measure, which may become a quasi-stationary distribution if exists and is summable.

Theorem 0.3. *For any given β satisfying $\rho \leq \beta < 1$, the m-th element of a β-invariant measure $\boldsymbol{\mu}^T(\beta)$ is given by the recursive formula:*

$$\boldsymbol{\mu}_m^T(\beta) = \sum_{j=1}^{m-1} \boldsymbol{\mu}_j^T(\beta)P_{j,m}(\beta\boldsymbol{I} - P_{m,m})^{-1}, \quad m \geq 2. \tag{6}$$

Finally, we focus on the domain of attraction of the β-invariant measure, which describe the criterion for the probability distribution $\boldsymbol{\nu}^T$ to satisfy the following equation:

$$\lim_{n\to\infty} P_{\boldsymbol{\nu}^T}(\Phi_n = (i,j)|T > n) = \boldsymbol{\mu}_j^T(\beta), \quad \text{for any } \; (i,j) \in E^*.$$

Theorem 0.4. *Let $\boldsymbol{M}^T = \{\boldsymbol{m}_{(i,j)}, (i,j) \in E^*\}$ be a measure satisfying*

$$\sum_{(i,j)\in E^*} \boldsymbol{m}_{(i,j)}\boldsymbol{\nu}_{(i,j)}(\rho) < \infty,$$

where $\boldsymbol{\nu}_{(i,j)}(\rho)$ is the elements of the ρ-invariant vector $\boldsymbol{\nu}_{(i,j)}(\rho)$. Suppose that $P(i,i) > 0$ for any $i \geq 1$ and $\mathbf{K} < \infty$. Then,

$$\lim_{n\to\infty} P_{\boldsymbol{M}^T}(\Phi_n = (i,j)|T > n) = \boldsymbol{\mu}_{(i,j)}^T(\rho), \quad \text{for any } \; (i,j) \in E^*.$$

Further Research

The above characterization of a quasi-stationary distribution is fundamental, but it is not complete without a full understanding of the long time behavior for the process, including the speed of convergence to the quasi-stationary distribution. We will investigate the speed of convergence to the quasi-stationary distribution for Markov-modulated Markov chains in the future.

References

[1] P.A. Ferrari, H. Kesten, S. Martinez, and S. Picco. Existence of quasistationary distributions. a renewal dynamical approach. *The Annals of Probability*, 23(2):501–521, 1995.

[2] M. Kijima. On the existence of quasi-stationary distributions in denumerable r-transient markov chains. *Journal of Applied Probability*, 496, 1992.

[3] A.G. Pakes. Quasi-stationary laws for markov processes: examples of an always proximate absorbing state. *Advances in Applied Probability*, 27:120–145, 1995.

[4] A.M. Yaglom. Certain limit theorems of the theory of branching processes. *Doklady Akademii Nauk SSSR*, 56:795–798, 1947.

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